Non · calculator !

7.2 Notes: Matrix Algebra

Matrices

Definition of Matrix

If m and n are positive integers, then an $m \times n$ (read "m by n") matrix is a rectangular array

$$\begin{array}{cc} & \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \cdots & \text{Column } n \end{array} \\ \begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

in which each **entry** a_{ij} of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

The entry in the ith row and jth column is denoted by the double subscript notation a_{ij}. For instance, a_{23} refers to the entry in the second row, third column. A matrix that has only one row is called a row matrix, and a matrix that has only one column is called a column matrix. A matrix having m rows and n columns is said to be of order m x n. If $m = n$, then the matrix is square of order m x m (or n x n). For a square matrix, the entries a_{11}, a_{22}, a_{33}, . . . are the main diagonal entries.

Example 1: Determine the order of each matrix. # of rows x # of columns

a. [2] 1×1

b. [-1 -3 0 ½] 1×4

c. $\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$ 2×2

d. $\begin{bmatrix} -4 & 1 \\ 5 & 0 \\ -3 & 2 \end{bmatrix}$ 3×2

Example 2: Solve for $a_{11}, a_{12}, a_{21}, and\ a_{22}$ in the following matrix equation.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

$a_{11} = 2$ $a_{12} = -1$

$a_{21} = -3$ $a_{22} = 0$

Definition of matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order m x n, then their sum is the m x n matrix given by

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of **different** orders is **undefined**. ☆

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an m x n matrix and c is a scalar, then the **scalar multiple** of A by c is the m x n matrix given by

$$cA = [ca_{ij}].$$

Example 3: For the following matrices, find (a) A + B, (b) A – B, (c) 3A, and (d) 3A – 2B.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

a) $A + B = \begin{bmatrix} 4 & 2 & 4 \\ -2 & -4 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

b) $A - B = \begin{bmatrix} 0 & 2 & 4 \\ -4 & 4 & -4 \\ 3 & -2 & 0 \end{bmatrix}$

c) $3A = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$

d) $\begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 2 & -8 & 6 \\ -2 & 6 & 4 \end{bmatrix}$

$= \begin{bmatrix} 2 & 6 & 12 \\ -11 & 8 & -9 \\ 8 & -3 & 2 \end{bmatrix}$

Example 4: Perform the indicated matrix operations.

$$3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right)$$

$3\begin{bmatrix} 2 & -2 \\ 7 & 8 \end{bmatrix}$

$= \begin{bmatrix} 6 & -6 \\ 21 & 24 \end{bmatrix}$

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Example 5: Solve for X in the equation 3X + A = B where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$$

$$3X + \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \left[\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right]$$

$$\frac{1}{3} \cdot 3X = \frac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} -4/3 & 2 \\ 2/3 & -2/3 \end{bmatrix}$$

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an m x n matrix and $B = [b_{ij}]$ is an n x p matrix, then the product AB is an m x p matrix

\# of Col of A = # of Rows of B

$$AB = [c_{ij}]$$

Where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$.

Example 6: Find the product AB using $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

3 x 2   2 x 2

ColofA = Row ofB

3 x 2

$$\begin{array}{r} 3-12 \\ -12+8 \\ -15+0 \end{array}\begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}\begin{array}{l} -2+3 \\ 8-2 \\ 10+0 \end{array}$$

Example: 7 Find the product AB and BA using $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}$ **and** $B = \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$.

2 x 3   Yes!   3 x 2   Yes   2 x 3

A

AB   2 x 2

$$\begin{array}{r} -2+0-3 \\ 4-1+2 \end{array}\begin{bmatrix} -5 & 7 \\ -3 & 6 \end{bmatrix}\begin{array}{l} 4+0+3 \\ 8+0-2 \end{array}$$

$$\begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}$$

BA   3 x 3

$$\begin{array}{cccc} & -2+8 & 0-4 & -6-8 \\ 1+0 & & & \\ -1+2 & & & \end{array}\begin{bmatrix} 6 & -4 & -14 \\ 1 & 0 & 3 \\ 1 & -1 & -5 \end{bmatrix}\begin{array}{l} \\ 3+0 \\ -3-2 \end{array}$$

0 - 1

Example 8: Find the product of each of the following or state that the product is not defined.

a. $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $=$ $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$

$2 \times 2 \quad 2 \times 2$

b. $\begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ $\begin{bmatrix} 10 \\ -5 \\ -9 \end{bmatrix}$

$3 \times 3 \quad 3 \times 1$

$6 + 4 + 0$

$3 - 2 - 6$

$1 + 8 \quad -18$

c. $\begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}$

$3 \times 2 \quad 3 \times 4$

$\neq$

product is not defined! for Matrix Mult.

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Definition of Identity Matrix

The n x n matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of order n x n** and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Note that an identity matrix must be *square*. When the order is understood to be n x n, you can denote I_n simply by I.

For example, $AI_n = A \text{ and } I_nA = A$.

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & 3 \end{bmatrix}.$$

The Inverse of a Square Matrix

Definition of the Inverse of a Square Matrix

Let $\boldsymbol{A}$ be an n x n matrix and let $\boldsymbol{I_n}$ be the n x n identity matrix. If there exists a matrix $\boldsymbol{A^{-1}}$ such that

$$\mathbf{AA^{-1}} = I_n = \mathbf{A^{-1}A}$$

Then $\mathbf{A^{-1}}$ is called the inverse of **A**. The symbol $\mathbf{A^{-1}}$ is read "*A inverse*."

Example 9: Show that B is the inverse of A, where

a) $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$ and B= $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$

$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$ and B= $\begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

If a matrix A has an inverse, then A is called Invertible or nonsingular; otherwise A is called singular. A nonsquare matrix <u>**cannot**</u> have an inverse.

The inverse of a 2 x 2 matrix: If A is a 2 x 2 matrix given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is given by:

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ The denominator $ad - bc$ is called the determinant of the 2 x 2 matrix A.

Example 10: If possible, find the inverse of each matrix

a) $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

$6 - 2 \neq 0$

4

$\frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$

b) $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

$6 - 6 = 0$

Inverse DNE

c) $A = \begin{bmatrix} 5 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$4(5a - c = 1)$

$3a + 4c = 0$

$+ \quad 20a - 4c = 4$

$23a = 4$

$a = \frac{4}{23}$

$5 \cdot \frac{4}{23} - c = 1$

$\frac{20}{23} - c = 1$

$-c = \frac{3}{23}$

$c = -3/23$

$4(5b - d = 0)$

$3b + 4d = 1$

$20b - 4d = 0$

$23b = 1$

$b = 1/23$

$5(\frac{1}{23}) - d = 0$

$\frac{5}{23} = d$

$A^{-1} = \begin{bmatrix} 4/23 & 1/23 \\ -3/23 & 5/23 \end{bmatrix}$